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Approaching the Singularity in Gowdy Universes

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Approaching the Singularity in Gowdy Universes

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Abstract

It has been shown that the cosmic censorship conjecture holds for polarized Gowdy spacetimes. In the more general, unpolarized case, however, the question remains open. It is known that cylindrically symmetric dust can collapse to form a naked singularity. Since Gowdy universes comprise gravitational waves that are locally cylindrically symmetric, perhaps these waves can collapse onto a symmetry axis and create a naked singularity. It is known that in the case of cylindrical symmetry, event horizons will not form under gravitational collapse, so the formation of a singularity on the symmetry axis would be a violation of the cosmic censorship conjecture.

To search for cosmic censorship violation in Gowdy spacetimes, we must have a better understanding of their singularities. It is known that far from the symmetry axes, the spacetimes are asymptotically velocity term dominated, but this property is not known to hold near the axes. In this thesis, we take the first steps toward understanding on and near axis behavior of Gowdy spacetimes with space-sections that have the topology of the three-sphere. Null geodesic behavior on the symmetry axes is studied, and it is found that in some cases, a photon will wrap around the universe infinitely many times on its way back toward the initial singularity.

Acknowledgements

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Contents

Introduction

The validity of the cosmic censorship conjecture, first proposed by Sir Roger Penrose in 1969, is among the most intriguing of the unsolved problems in Einstein's general theory of Relativity. The conjecture states that singularities (other than the initial "big bang" and the final "big crunch" singularities) are hidden within event horizons (i.e. within black holes). This is actually a form of the weak cosmic censorship conjecture. The strong cosmic censorship conjecture asserts that an observer who falls into the black hole will still not see the singularity. In essence, it is the statement that naked singularities do not exist.[1] The problem of cosmic censorship is not well-posed. In fact, spacetimes with naked singularities have been constructed, but they are generally dismissed as being physically unreasonable. For example, cylindrically symmetric dust can collapse to form a naked singularity, but the dust particles must have zero size (sometimes called mathematical dust or perfect dust).[2] Counterexamples lead to a better understanding of the problem itself, and the more physically relevant the spacetime, the more interesting the outcome.

A necessary step in the direction of validating the cosmic censorship conjecture is understanding the global properties of known solutions to Einstein's field equations. An important global property is causal structure; the labeling of events as "to the past of," "to the future of," and "simultaneous with" any given event in the spacetime.[3]

In this thesis, we investigate the causal structure and light-cone topology of Gowdy universes whose space-sections have the topology of the three-sphere (S^3). These spacetimes are a subset of the most complex exact solutions known.

In Einstein's Relativity, freely falling particles follow geodesics in spacetime. To probe causal structure, we investigate geodesic behavior near the initial and final singularities. Gowdy spacetimes comprise gravitational waves that are locally cylindrical near two symmetry axes.[4] Gravitational waves have two polarizations.[5] In this thesis, we consider only polarized Gowdy universes, those in which one polarization has been turned off. Far from the symmetry axes, the polarized Gowdy spacetimes have been shown to be asymptotically velocity-term dominated near the singularities (AVTDS).[6] This is the property that near the singularity, the gravitational physics is the same as if all spatial derivatives were dropped from the metric functions.

In chapter 1 is a review of the fundamental mathematical tools used in the general theory of Relativity. The main focus is on those concepts necessary to understand the work in later chapters.

In chapter 2, we discuss the linearized theory of gravitation. The derivations in this chapter closely follow those of Misner, Thorne, and Wheeler in *Gravitation.*[5] The linearized theory is a weak-field approximation and is in excellent agreement with experiment in the absence of very large gravitational fields.

The linearized theory leads directly to gravitational waves, which we discuss in chapter 3.

We discuss Gowdy universes in chapter 4. These universes are closed, vacuum spacetimes that are inhomogeneous due to the presence of gravitational waves.

The cosmic censorship conjecture and asymptotic behavior of the Gowdy metric are discussed in chapter 5, and in chapter 6, our research on geodesic behavior in Gowdy S^3 spacetimes is presented.

Chapter 1
Mathematics in Curved Spacetime

In this thesis, we investigate geodesic behavior near the singularities in Gowdy spacetimes with space-sections diffeomorphic to the three-sphere (S^3). The purpose of this section is to introduce the mathematical tools and notation used throughout this work. We begin with the concept of coordinates, and work our way up to geodesics in a spacetime.

1.1 Coordinates

In three dimensional space, a moving particle traces out a path. In four dimensional spacetime, this path is called a worldline. When two particles collide, their worldlines meet at a point, which is called an event. To locate an event, follow the worldlines of the particles involved. If each event in the spacetime is given a name for reference, every point can be uniquely identified without the need for coordinates. Coordinates, however, are useful. How else does one know in which direction to travel along the worldlines? Coordinates allow one to order the events. It is important to note that the coordinates are merely a convenience and have no physical significance. Much the way room numbers in a building allow a visitor to efficiently locate a room but give no indication of the distance between the rooms, coordinates allow one to efficiently locate an event but give no indication of the distance between two events. The lesson: The choice of a system of coordinates is arbitrary.

Often a single set of coordinates is not sufficient to locate every event in a spacetime. Consider the surface of a sphere. A stereographic projection maps points on the surface onto the real plane.[7] However, one point, the north pole, does not map to the plane. We can, however, join coordinate systems by ensuring they are consistent in the region of overlap.[1] The surface of the sphere requires at least two coordinate patches. One way to create two patches is to apply the stereographic projection function to two different planes as in Figure 1.1.

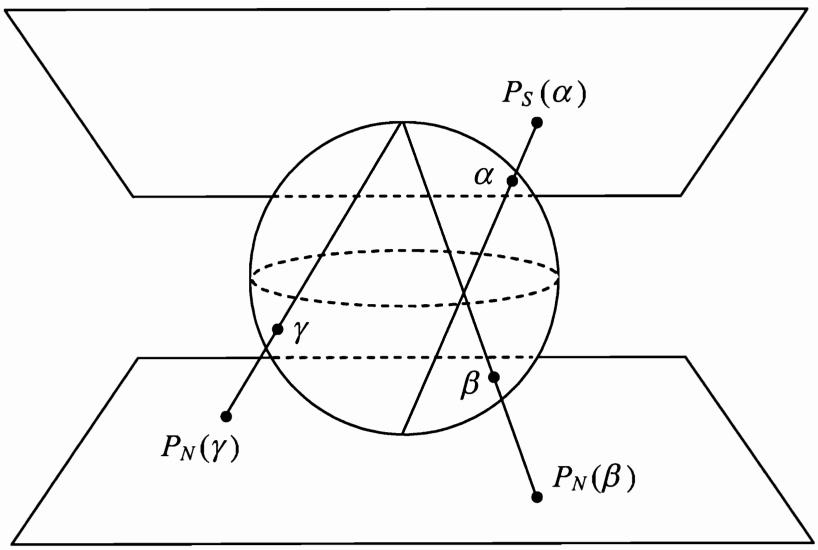

Fig. 1.1. Two stereographic projections of the surface of a two sphere; one for the upper hemishphere and one for the lower hemishphere.

The use of different coordinate systems necessitates a transformation law for coordinates.[8] Consider two coordinate systems; (η, ζ) and (x, y). It may be helpful to think of these as two descriptions of the Euclidean plane with (x, y) being the standard

[1] This idea will be made more precise in the section on manifolds.

description (a square grid) and (η, ζ) being a set of generalized coordinates. The lines of constant η or of constant ζ need not be parallel, and the angles between intersecting coordinate lines need not be perpendicular. In this two-dimensional example, the local form of the transformation from one set of coordinates to the other is given by

$$\begin{bmatrix} d\eta \\ d\zeta \end{bmatrix} = \begin{bmatrix} \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \tag{1}$$

This relation may be interpreted as transforming small changes in one set of coordinates to small changes in the other set.

The matrix

$$\left[\Lambda^{\alpha'}{}_{\beta}\right] = \begin{bmatrix} \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial y} \end{bmatrix} \tag{2}$$

is known as the transformation matrix. The determinant of $\left[\Lambda^{\alpha'}{}_{\beta}\right]$ is the Jacobian of the coordinate transformation. If the Jacobian vanishes at a point, the coordinates are singular there.

1.2 Vectors and Forms

The concept of a vector space is familiar from elementary linear algebra. In the discussion that follows, we consider vector spaces that have the same dimensionality as the spacetimes considered in General Relativity, namely four.

Recall that a vector, v, may be expanded in a basis, $\{e_0, e_1, e_2, e_3\}$, as

$$v = v^0 e_0 + v^1 e_1 + v^2 e_2 + v^3 e_3, \tag{3}$$

where the v^μ are the components of v in the basis $\{e_\mu\}$. Using the Einstein summation convention, this may be written

$$v = v^\mu e_\mu. \tag{4}$$

Greek characters take values from the set $\{0, 1, 2, 3\}$. The zeroth component will always be the time component. We may wish to separate the time component and the space components;

$$v = v^0 e_0 + v^i e_i. \tag{5}$$

Latin characters take values from the set $\{1, 2, 3\}$.

In flat space physics, vectors are thought of as bilocal geometric objects (objects with a head and a tail). The geometric viewpoint is an extremely useful one in curved space physics, but bilocality poses some problems. We require a local, i.e. at a point, definition of vector.[5]

There is a one-to-one correspondence between vectors and partial derivative operators.[3] That partial derivative operators form a vector space is easily verified, and these operators are local objects, so they meet our need of a local definition of vector. We define the vector basis

$$e_\mu = \frac{\partial}{\partial x^\mu} \tag{6}$$

where x^μ are the coordinate functions and $\frac{\partial}{\partial x^\mu}$ are partial derivative operators. A vector, then, may be written

$$v = v^\mu \frac{\partial}{\partial x^\mu}. \tag{7}$$

It is convenient to think of a vector as a mapping from scalar fields to real (or complex) numbers. Allowing the vector, v, to act on a scalar function, f, we get

$$vf = v^{\mu}\frac{\partial f}{\partial x^{\mu}}. \tag{8}$$

Thus, v is just a directional derivative. Directional derivatives are evaluated at a point in the spacetime, and the basis $\{\frac{\partial}{\partial x^{\mu}}\}$ spans a vector space known as the tangent space at that point. Each point in the spacetime will have a different tangent space.

The space dual to the tangent space is called the cotangent space or the space of one-forms. We define the basis of the cotangent space $\{\omega^{\nu}\}$ such that

$$\omega^{\nu}\cdot e_{\mu} = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases} \tag{9}$$

where $\{e_{\mu}\}$ is the basis of the tangent space. The basis of the cotangent space dual to the tangent space with holonomic basis $\{\frac{\partial}{\partial x^{\mu}}\}$ is $\{dx^{\nu}\}$. The space of one-forms is a vector space, hence we can expand a form, α, in a basis;

$$\alpha = \alpha_{\nu}dx^{\mu}. \tag{10}$$

Notice that

$$\alpha\cdot v = \alpha_{\nu}v^{\mu}. \tag{11}$$

We define the function

$$\alpha(v) = \alpha\cdot v = v\cdot\alpha = v(\alpha) \tag{12}$$

so that a form is a real (or complex) valued function of vectors, and a vector is a real (or complex) valued function of forms.

One may consider the space dual to the dual space; the double dual. For finite dimensional spaces, however, there is a natural isomorphism between the double dual and the tangent space. We therefore identify the double dual as the tangent space and refer to it no longer.[3]

1.3 Tensors

Vectors and forms, thus defined, are special cases of a more general geometric object called a tensor. Tensors are multilinear functions of vectors and forms into real (or complex) numbers. For tangent space V and cotangent space V^* we have

$$T : \underbrace{V^* \times \cdots \times V^*}_{n} \times \underbrace{V \times \cdots \times V}_{m} \longrightarrow \mathbb{R}. \tag{13}$$

For n forms and m vectors, the tensor is said to be of type (n, m) and have covariant rank n and contravariant rank m. As examples, a vector is a contravariant tensor of rank one, and a form is a covariant tensor of rank one.

The components of a tensor in a given basis are found by dropping basis objects into the slots of the tensor;

$$T\left(\omega^{\alpha_1}, \ldots, \omega^{\alpha_n}, e_{\beta_1}, \ldots, e_{\beta_m}\right) = T^{\alpha_1 \ldots \alpha_n}{}_{\beta_1 \ldots \beta_m}. \tag{14}$$

For example, the components of a vector are

$$v^\nu = v\left(\omega^\nu\right). \tag{15}$$

Two important operations on tensors are contraction and outer product.[9]

Contraction may be thought of as a mapping from tensors of type (n, m) to tensors of type $(n-1, m-1)$. Contraction on the i^{th} (covariant) and j^{th} (contravariant) slots is given by

$$T\left(\omega^{\alpha_1},\ldots,\omega^{\gamma},\ldots,\omega^{\alpha_n},e_{\beta_1},\ldots,e_{\gamma},\ldots,e_{\beta_m}\right)=T^{\alpha_1\ldots\gamma\ldots\alpha_n}{}_{\beta_1\ldots\gamma\ldots\beta_m}. \tag{16}$$

Recall that like indices, where one is up and one is down, are summed over. The contraction of a tensor of type $(1,1)$ is called the trace of the tensor and yields a scalar. We see that contraction is one tool for forming a new tensor out of old ones. It is important to note that tensor contraction requires a set of basis vectors for its definition, but the operation is invariant under any change of basis.

Another tool for forming a new tensor out of old ones is the outer product. The outer product of a tensor, T, of type (n,m) with a tensor, T', of type (k,l) yields a tensor, $T\otimes T'$, of type $(n+k,m+l)$. Every tensor may be written as a sum of outer products of vectors and forms;

$$T=T^{\alpha_1\ldots\alpha_n}{}_{\beta_1\ldots\beta_m}e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_n}\otimes\omega^{\beta_1}\otimes\cdots\otimes\omega^{\beta_m}. \tag{17}$$

1.4 The Metric Tensor

A very important tensor is the metric tensor. A metric tensor g is a rank two covariant tensor that defines length in a metric space. In other words, g maps pairs of vectors into real numbers, and if the two vectors fed into the metric are equal, the resulting number is the length of the vector. This mapping is the inner product;

$$g\left(u,v\right)=u\cdot v \tag{18}$$

where, for vector basis $\{e_\mu\}$,

$$u \cdot v = u^\alpha v^\beta g\left(e_\alpha, e_\beta\right) = u^\alpha v^\beta g_{\alpha\beta}. \tag{19}$$

Thus, the inner product depends on the components of the vectors and the components of the metric tensor. The metric tensor is symmetric, and we have

$$u \cdot v = v \cdot u. \tag{20}$$

As an example, the flat space metric is given by

$$g = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{21}$$

and the inner product of vectors u and v is

$$-u^0 v^0 + u^1 v^1 + u^2 v^2 + u^3 v^3. \tag{22}$$

In a coordinate basis, the metric tensor may be written as

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu. \tag{23}$$

Often this is written as the squared, infinitesimal distance

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{24}$$

where the outer product of basis forms is assumed but not explicitly stated.

The metric may be used to raise or lower indexes on tensors. An index is lowered according to the rule

$$T_\mu = g_{\mu\nu} T^\nu. \tag{25}$$

The metric is invertible, and so multiplying both sides of 25 by the inverse metric

$$g^{-1} = g^{\mu\nu} \tag{26}$$

yields a formula for raising an index:

$$T^{\nu} = g^{\mu\nu} T_{\mu}. \tag{27}$$

A special notation is often used to indicate the metric that describes the flat spacetime of Special Relativity. In this thesis, we use the greek letter η for this metric. The components of η are given by 21. In the general theory of Relativity, the geometry of the spacetime is flat for sufficiently small neighborhoods of each event. The metric η is sometimes called the Minkowski metric, and we say that spacetime is locally Minkowskian.

1.5 Manifolds

Our intuitive idea of space is made mathematically precise by the notion of a manifold.[10] An n-dimensional manifold is a topological space whose local structure is like $\mathbb{R}^n$ but whose global structure may be different from $\mathbb{R}^n$. It is a space that may be curved over an extended region but is flat for a very small region at each point. The surface of a sphere, called a two-sphere, is an example of a 2-dimensional manifold. The regular Euclidean space used in Newtonian mechanics is a 3-dimensional manifold that is globally flat. In General Relativity, spacetime is a 4-dimensional manifold.

A more formal definition of manifold requires some terminology. An open subset U_i of an n-dimensional topological space $\mathcal{M}$ along with a mapping $\psi_i : U_i \rightarrow \mathbb{R}^n$ is a chart. The transition function $\psi_i \circ \psi_j^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ relates overlapping maps (see Figure 1.2).A collection of pairs (U_i, ψ_i) such that the U_i cover $\mathcal{M}$, and the

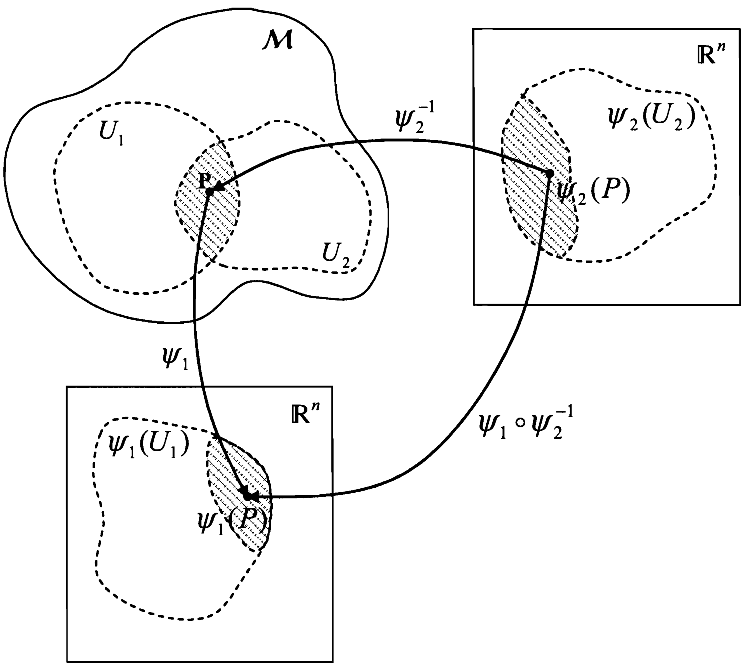

Fig. 1.2. Coordinate functions associate n-tuples with each point in a manifold. Transition functions relate n-tuples in overlapping coordinate patches.

transition functions are of class C^p, is a C^p atlas on $\mathcal{M}$. A C^p manifold is the set $\mathcal{M}$ along with a C^p atlas on $\mathcal{M}$. The mapping ψ_i is called a coordinate function. Each chart defines a local coordinate system (or coordinate patch) on $\mathcal{M}$. In other words, a manifold is a set along with a complete set of appropriately continuous and differentiable overlapping coordinate patches.

A curve is a function that maps real numbers into points in the manifold;

$$C : \mathbb{R} \rightarrow \mathcal{M}. \tag{28}$$

Each real number, λ, is called a curve parameter. If a coordinate basis is introduced, we get a function from the real numbers into Euclidean n-space;

$$(\psi \circ C)(\lambda) = x^1(\lambda)\, e_1 + x^2(\lambda)\, e_2 + \cdots + x^n(\lambda)\, e_n. \tag{29}$$

At each point in the manifold, thereby at each point on the curve, is a tangent space. Therefore, we may operate on the function with a vector in the tangent space;

$$v(\psi \circ C) = v^\mu \frac{\partial(\psi \circ C)}{\partial x^\mu}. \tag{30}$$

We may define a function on the manifold;

$$f : \mathcal{M} \to \mathbb{R}. \tag{31}$$

The composite function

$$f \circ C : \mathbb{R} \to \mathbb{R} \tag{32}$$

gives values for the function evaluated along the curve.

Consider the derivative

$$\frac{df}{d\lambda} = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{d\lambda}. \tag{33}$$

This derivative gives the rate of change of the function along the curve with parameter λ. This can be rewritten

$$\frac{d}{d\lambda}f = \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu}f, \tag{34}$$

which implies that

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu}. \tag{35}$$

Notice this is a vector in the tangent space with components $\frac{dx^\mu}{d\lambda}$. We define this vector to be the tangent vector to the curve. If this curve is the worldline of some particle and has proper time, τ, as its curve parameter, we call $\frac{d}{d\tau}$ the four-velocity of the particle.

1.6 Curvature

Our intuitive notion of curvature comes from surfaces embedded in higher dimensional space. For example, the two-dimensional surface of a sphere looks curved in $\mathbb{R}^3$. This notion is embodied in the mathematical concept of extrinsic curvature. In Einstein's Relativity, we do not consider spacetime to be embedded in higher dimensional flat space. In other words, the four-dimensional manifold we use to model spacetime is not embedded in $\mathbb{R}^5$. Therefore, we wish to define curvature as an intrinsic property of the manifold itself. Intrinsic curvature is defined in terms of parallel transport.[3]

To parallel transport a vector is to move the vector along some curve while keeping the vector tangent to the manifold such that the vector always points in the same direction. This process is fairly intuitive in the Euclidean plane. If we parallel transport a vector around some closed path, the vector in its final position points in the same direction as it did in its initial position (and anywhere else along the curve for that matter). But if we parallel transport a vector around a closed path on the surface of a sphere, the vector in its final position may point in a different direction than it did in its initial position. Furthermore, different paths may produce different final vectors.

For our intuitive idea of parallel transport to be made precise, we must have a way of comparing vectors in different vector spaces. Given only the structure of a manifold, this is not possible. The additional structure we require is called a connection. In this section, we define the connection and the covariant derivative operator. We use these to define parallel transport, and we use parallel transport (in particular, the path dependence) to define the curvature tensor. Finally, geodesics are defined.

The covariant derivative operator, D, is a map that creates a tensor field of type $(n, m+1)$from a tensor field of type (n, m). The operator, D, obeys the following algebraic rules.[2] The subscript on the operator D denotes the direction in which the derivative is taken and does not indicate the components of D.

1. Leibnitz rule: For any scalar fields f and g, vector fields u and v, and tensor fields T_1 and T_2,

$$D_v (fT_1 + gT_2) = D_v fT_1 + D_v gT_2 \tag{36}$$

$$= (D_v f) T_1 + f D_v T_1 + (D_v g) T_2 + g D_v T_2. \tag{37}$$

2. Additivity: For any scalar fields f and g, vector fields u and v, and tensor field T,

$$D_{fu+gv} T = f D_u T + g D_v T. \tag{38}$$

The rules for differentiating a field allow us to compare values of the field at neighboring points on the manifold. An operation of this sort is called a connection. In addition to the algebraic properties above, Einstein's general theory of Relativity

[2] The following discussion stems from the unpublished notes of Robert H. Gowdy at Virginia Commonwealth University.

requires the derivative operator to be torsion free[3], i.e., for any scalar field f and vector fields u and v,

$$D_u D_v f = D_v D_u f. \tag{39}$$

We define the action of the derivative on a scalar field by

$$D_u f \equiv uf. \tag{40}$$

Using equations 39 and 40, the commutator of two vector fields can be derived;

$$[u, v] = D_u v - D_v u. \tag{41}$$

When a holonomic basis is introduced, we can define the coordinates of the covariant derivative operator. As an example, we can expand $D_u v$ as

$$D_u v = D_{u^\alpha e_\alpha} v^\beta e_\beta = u^\alpha D_{e_\alpha} v^\beta e_\beta. \tag{42}$$

The v^β are just scalar functions so we have

$$D_u v = u^\alpha \left(D_{e_\alpha} v^\beta\right) e_\beta + u^\alpha v^\beta D_{e_\alpha} e_\beta \tag{43}$$

$$= u^\alpha \left(\frac{\partial}{\partial x^\alpha} v^\beta\right) e_\beta + u^\alpha v^\beta D_{e_\alpha} e_\beta. \tag{44}$$

The covariant derivative acts on basis vectors according to

$$D_{e_\alpha} e_\beta \equiv \Gamma^\mu{}_{\beta\alpha} e_\mu \tag{45}$$

where the $\Gamma^\mu{}_{\beta\alpha}$ are called connection coefficients. The subscripts are ordered so that the first lowered index is the index of basis vector field being differentiated, and the second lowered index indicates the direction. We now have, for the covariant

[3] It should be noted that some theories of gravitation, such as the Brans-Dicke theory, do not require zero torsion.

derivative of the vector field v in the direction of the vector field u,

$$D_u v = \left(u^\alpha \frac{\partial v^\beta}{\partial x^\alpha} + u^\alpha v^\mu \Gamma^\beta{}_{\mu\alpha} \right) e_\beta \tag{46}$$

where dummy indexes have been renamed to get the components of $D_u v$;

$$(D_u v)^\beta = u^\alpha \frac{\partial v^\beta}{\partial x^\alpha} + u^\alpha v^\mu \Gamma^\beta{}_{\mu\alpha}. \tag{47}$$

It is common to use a semicolon to denote a covariant derivative and a comma to denote a partial derivative. In this notation, we can write $D_u v$ as follows:

$$D_u v = u^\alpha D_{e_\alpha} v \tag{48}$$

$$\equiv u^\alpha v^\beta{}_{;\alpha} e_\beta \tag{49}$$

$$= \left(u^\alpha v^\beta{}_{,\alpha} + u^\alpha v^\mu \Gamma^\beta{}_{\mu\alpha} \right) e_\beta \tag{50}$$

The connection coefficients can be determined from the metric components via the equation

$$\Gamma^\mu{}_{\beta\alpha} = \frac{1}{2} g^{\mu\nu} \left(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\beta\alpha,\nu} \right). \tag{51}$$

Given a curve, C, with tangent vector u, a vector v defined along the curve is said to be parallel transported if

$$D_u v = 0. \tag{52}$$

In other words, a vector is parallel transported along a curve if there is no twisting or turning of the vector. It should be noted that equation 52 is true in general only for an affine connection. An important special case of 52 occurs when the differentiated vector is the tangent vector to the curve:

$$D_u u = 0. \tag{53}$$

Equation 53 is called a geodesic equation, and a curve with this property is called a geodesic. Geodesics are the generalization of straight lines to a curved space. The great circles on Earth provide simple examples of geodesics on a curved surface. Locally these circles look flat. If one walks toward the North Pole in a line that is as straight as is possible, that person is walking along a great circle. In General Relativity, an object in free fall follows a geodesic of spacetime.

Chapter 2
The Linearized Theory

Herein, we follow the derivation given by Misner, Thorne, and Wheeler.[5]

For weak gravitational fields, the metric can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{54}$$

where η is the Minkowski metric, and $|h_{\mu\nu}| \ll 1$. Substituting the right hand side of 54 into the equation for the connection coefficients (equation 51) yields

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2}\left(\eta^{\alpha\beta} + h^{\alpha\beta}\right)\left[\left(\eta_{\beta\mu,\nu} + h_{\beta\mu,\nu}\right) + \left(\eta_{\beta\nu,\mu} + h_{\beta\nu,\mu}\right) - \left(\eta_{\mu\nu,\beta} + h_{\mu\nu,\beta}\right)\right]. \tag{55}$$

The derivatives of the flat space metric vanish. Eliminating terms non-linear in $h_{\mu\nu}$ yields

$$\begin{aligned}\Gamma^{\alpha}{}_{\mu\nu} &= \frac{1}{2}\eta^{\alpha\beta}\left(h_{\beta\mu,\nu} + h_{\beta\nu,\mu} - h_{\mu\nu,\beta}\right) &(56)\\ &\equiv \frac{1}{2}\left(h^{\alpha}{}_{\mu,\nu} + h^{\alpha}{}_{\nu,\mu} - h_{\mu\nu,}{}^{\alpha}\right). &(57)\end{aligned}$$

Using these connection coefficients, the components of the Ricci tensor become

$$\begin{aligned}R_{\mu\nu} &= \Gamma^{\alpha}{}_{\mu\nu,\alpha} - \Gamma^{\alpha}{}_{\mu\alpha,\nu} &(58)\\ &= \frac{1}{2}\left(h^{\alpha}{}_{\mu,\nu\alpha} + h^{\alpha}{}_{\nu,\mu\alpha} - h_{\mu\nu,}{}^{\alpha}{}_{\alpha}\right) - \frac{1}{2}\left(h^{\alpha}{}_{\mu,\alpha\nu} + h^{\alpha}{}_{\alpha,\mu\nu} - h_{\mu\alpha,}{}^{\alpha}{}_{\nu}\right) &(59)\\ &= \frac{1}{2}\left(h^{\alpha}{}_{\nu,\mu\alpha} - h_{\mu\nu,}{}^{\alpha}{}_{\alpha} - h^{\alpha}{}_{\alpha,\mu\nu} + h_{\mu\alpha,}{}^{\alpha}{}_{\nu}\right) &(60)\end{aligned}$$

where terms non-linear in $h_{\mu\nu}$ have been dropped. Contraction of the linearized Ricci tensor yields the linearized scalar curvature;

$$R \equiv g^{\mu\nu}R_{\mu\nu} \approx \eta^{\mu\nu}R_{\mu\nu}. \tag{61}$$

We get the Einstein tensor from the Ricci tensor and the scalar curvature. To simplify the results, we make the following definition:

$$\bar{S}_{\mu\nu} \equiv S_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}S \tag{62}$$

where $S_{\mu\nu}$ are the components of any symmetric tensor, and $S \equiv S^{\mu}{}_{\mu}$. With this definition, the components of the linearized Einstein tensor are, to first order in $h_{\mu\nu}$,

$$G_{\mu\nu} = \bar{R}_{\mu\nu}. \tag{63}$$

The linearized field equations are then

$$\bar{R}_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{64}$$

In terms of the perturbation metric, h, the field equations are

$$-\bar{h}_{\mu\nu,}{}^{\alpha}{}_{\alpha} - \eta_{\mu\nu}\bar{h}_{\alpha\beta,}{}^{\alpha\beta} + \bar{h}^{\alpha}{}_{\nu,\mu\alpha} + \bar{h}_{\mu\alpha,}{}^{\alpha}{}_{\nu} = 16\pi T_{\mu\nu}. \tag{65}$$

Applying the Lorentz gauge,

$$\bar{h}^{\mu\nu}{}_{,\nu} = 0, \tag{66}$$

greatly simplifies the field equations, and we get

$$-\bar{h}_{\mu\nu,}{}^{\alpha}{}_{\alpha} = 16\pi T_{\mu\nu} \tag{67}$$

where $\bar{h}_{\mu\nu,}{}^{\alpha}{}_{\alpha} = \Box\bar{h}_{\mu\nu}$, and $\Box$ is the d'Alembertian. The vacuum field equations are then just the familiar wave equations

$$\Box\bar{h}_{\mu\nu} = 0. \tag{68}$$

Chapter 3
Gravitational Waves

One solution for $\Box\bar{h}_{\mu\nu} = 0$ is the plane-wave solution;

$$\bar{h}_{\mu\nu} = \mathrm{Re}\left\{\bar{\alpha}_{\mu\nu}e^{ik_\alpha x^\alpha}\right\} \tag{69}$$

where

$$k \cdot k = 0. \tag{70}$$

Recall that the field equations take the form of 68 in the Lorentz gauge. Thus, we are led to impose the transverse wave condition, equation 66, which yields

$$\bar{\alpha}^{\mu\nu}k_\nu = 0. \tag{71}$$

The amplitude is symmetric and, thereby, has, at most, ten independent components. Four components are fixed by imposition of the transverse wave condition. Most of the remaining arbitrariness comes from the fact that the Lorentz gauge does not fully fix the coordinates. We have the gauge freedom

$$\bar{h}_{\mu\nu} \longrightarrow \bar{h}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\xi^\rho{}_{,\rho} \tag{72}$$

where

$$\Box\xi_\mu = 0. \tag{73}$$

Much of the ambiguity can be removed by imposing the condition

$$\bar{\alpha}^{\mu 0} = 0. \tag{74}$$

These are three additional constraints (they may appear to be four, but $k_\mu \bar{a}^{\mu 0} = 0$ is already satisfied). The last bit of coordinate freedom can be removed by imposing the condition

$$\sum_{i=1}^{3} \bar{a}^{ii} = 0. \tag{75}$$

The three conditions 71, 74, and 75 are known as the radiation gauge (sometimes called the transverse-traceless gauge).[4] Imposition of this gauge restricts the possible amplitudes, which, in matrix form, are

$$[\bar{a}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{76}$$

assuming $\frac{\partial}{\partial x^3}$ is the direction of propagation. We are left with two degrees of freedom, which represent the two polarizations of the plane-wave.

While plane gravitational waves are allowed in the linearized theory, it was not clear that the full, nonlinear theory would lead to solutions allowing gravitational radiation. The first exact solutions of Einstein's field equations that did not assume a weak gravitational field were found by Einstein and Rosen. These solutions include the cylindrically symmetric waves that play a role in Gowdy universes.

[4] The particular forms of the gauge conditions used here were given in the unpublished notes of Robert H. Gowdy.

Chapter 4
Gowdy Universes

Gowdy universes are the most complex exact solutions to Einstein's field equations known. Thus, they make very good test solutions for new formalisms in Relativity as well as for new computer software designed to study the evolution of spacetimes.[11] Gowdy universes have boundary conditions that are acceptable in the standard model of cosmology. In fact, those whose space-sections are three-spheres have the same topology as a family of Friedmann-Robertson-Walker solutions, which are part of the standard model.[3] Because of this, Gowdy universes are often used as toy cosmological models to study such phenomena as particle creation in the early universe.[12] In this chapter, we investigate these solutions. Our main focus is on the three-sphere solutions, so after a brief general introduction, we make this specialization.

Solutions to Einstein's equation with two parameter isometry groups are called G_2 spacetimes. Gowdy universes are non-stationary G_2 spacetimes with at least two spacelike Killing vectors and compact spacelike hypersurfaces of finite volume.[4] Exact solutions can be found for the vacuum field equations, and these spacetimes are inhomogeneous due to the presence of gravitational waves. Locally, these waves are just the Einstein-Rosen cylindrical waves.

The spacelike hypersurfaces of a Gowdy universe have a topology homeomorphic to one of three topologies; the three-torus (T^3), the three-handle, also called the wormhole, $(S^1 \times S^2)$, or the three-sphere (S^3). Solutions with $(0,1) \times S^1 \times S^2$ and $(0,1) \times S^3$ topologies have degenerate isometry group trajectories, whereas those with the three-torus topology do not. This thesis is restricted to Gowdy spacetimes with $(0,1) \times S^3$ topology and reflection symmetry of one of the group coordinates, which serves to polarize the gravitational waves.[13]

The group orbits are compact and intrinsically flat. We may choose the group parameters to be angles, σ and δ, that range from 0 to 2π. If the trajectories are labeled by θ and t, the Gowdy metric for the $(0,1) \times S^1 \times S^2$ and $(0,1) \times S^3$ topologies with a reflection symmetry of one of the group coordinates has the form

$$ds^2 = e^{2a}\left(d\theta^2 - dt^2\right) + R\left(e^{2\psi} d\sigma^2 + e^{-2\psi} d\delta^2\right) \tag{77}$$

where a, ψ, and R depend only on θ and t.[14]

The independent Einstein field equations are

$$4R\dot{a}\left(\dot{R}^2 - R'^2\right) = 4R^2\dot{R}\left(\dot{\psi}^2 + \psi'^2\right) + \dot{R}R'^2 - \dot{R}^3 + 4\dot{R}\ddot{R}R - 4R\dot{R}'R' - 8R^2\dot{\psi}\psi'R' \tag{78}$$

$$4Ra'\left(\dot{R}^2 - R'^2\right) = -4R^2R'\left(\dot{\psi}^2 + \psi'^2\right) - R'\dot{R}^2 + R'^3 - 4R'\ddot{R}R + 4R\dot{R}'\dot{R} + 8R^2\dot{\psi}\psi'\dot{R} \tag{79}$$

$$\left(\dot{R}\dot{\psi} + R\ddot{\psi}\right) - \left(R'\psi' + R\psi''\right) = 0 \tag{80}$$

$$\ddot{R} - R'' = 0 \tag{81}$$

where the primes denote partial derivatives with respect to θ, and the dots denote partial derivatives with respect to t. Equations 80 and 81 are simply wave equations. The solutions of these equations determine the topology. Coordinates can be found such that

$$R = \sin\theta \sin t \tag{82}$$

which solves equation 81. This solution corresponds to spacelike hypersurfaces with $S^1 \times S^2$ and S^3 topologies. In this thesis, only solutions with S^3 space sections will be considered. The solution for equation 80 for the S^3 topology is

$$\psi = W - \frac{1}{2} \ln \tan \left(\frac{\theta}{2}\right) \tag{83}$$

where

$$W = \sum_j \left[A_j P_j (\cos t) + C_j Q_j (\cos t)\right] P_j (\cos\theta), \tag{84}$$

and P_j and Q_j are Legendre polynomials of the first and second kind, respectively.[4]

The coordinates constructed above (sometimes called Gowdy coordinates) are globally defined, and Figure 4.3 shows the coordinates for the three-sphere topology. To ensure the geometry is regular, however, certain conditions must be satisfied. In

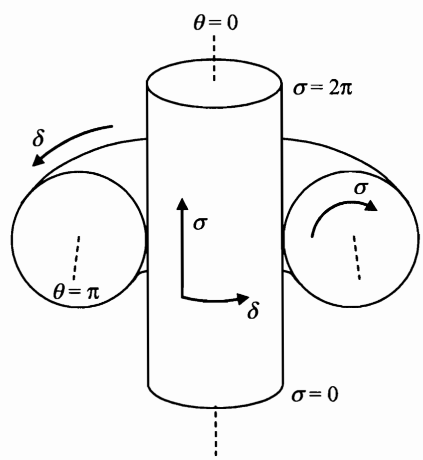

Fig. 4.3. Gowdy coordinate patches for a three-sphere sliced along $\theta = \pi/2$. The two-torus around the waist of the cylinder has been cut in half so the coordinates can be seen. Note that the top and bottom of the cylinder are identified and this is also a two-torus.

terms of canonical coordinates, the area function, R, changes character in different regions of the spacetime. The level curves of R are depicted in Figure 4.4, where $u = t - \theta$ and $v = t + \theta$ are, respectively, retarded time and advanced time coordinates.[4] We can see the gradient of R is either spacelike, lightlike, or timelike depending on

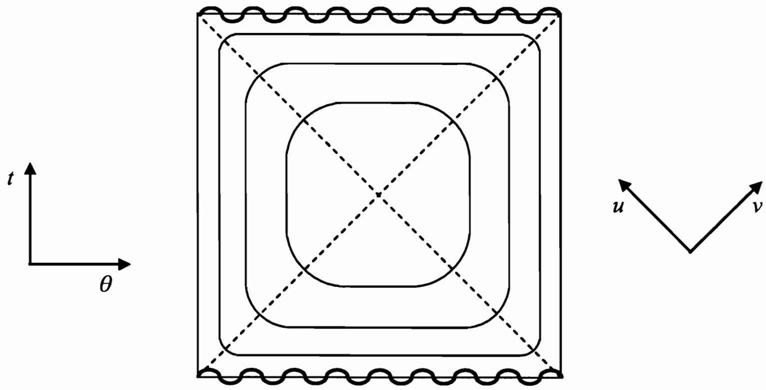

Fig. 4.4. Level R curves.

which region is considered. The independent Einstein equations can be written in terms of these coordinates:[13]

$$R_+ a_+ = R\psi_+^2 + \frac{1}{2}R_{++} - \frac{1}{4}R\left(R_+/R\right)^2 \tag{85}$$

$$R_- a_- = R\psi_-^2 + \frac{1}{2}R_{--} - \frac{1}{4}R\left(R_-/R\right)^2 \tag{86}$$

$$\left(\dot{R}\dot{\psi} + R\ddot{\psi}\right) - \left(R'\psi' + R\psi''\right) = 0 \tag{87}$$

$$R_{+-} = 0 \tag{88}$$

where f_+ denotes partial derivatives with respect to advanced time, and f_- denotes partial derivatives with respect to retarded time. Given that $R = \sin\theta \sin t$, $R_+ = 0$ on the null surface $v = \pi$, and $R_- = 0$ on the null surface $u = 0$. In terms of canonical Einstein-Rosen coordinates, R and $T = \cos\theta\cos t$, each of the four regions has a different metric that must be pieced together along the null surfaces $u = 0$ and $v = \pi$. The matching constraints are of one of two types:

$$\text{Type I:} \qquad \frac{\partial\psi}{\partial t} = 0 \text{ and } \frac{\partial\psi}{\partial\theta} = \pm 1 \tag{89}$$

$$\text{Type II:} \qquad \frac{\partial\psi}{\partial t} = \pm 1 \text{ and } \frac{\partial\psi}{\partial\theta} = 0 \tag{90}$$

A bit of calculation with the Legendre function expansion of the wave function ψ reveals that these constraints impose very simple restrictions on the wave amplitudes:

$$\text{Type I:} \qquad \sum_{n=0}^{\infty} C_{2n} = 0 \text{ and } \sum_{n=0}^{\infty} C_{2n+1} = s + \frac{1}{2} \tag{91}$$

$$\text{Type II:} \qquad \sum_{n=0}^{\infty} C_{2n} = s \text{ and } \sum_{n=0}^{\infty} C_{2n+1} = \frac{1}{2} \tag{92}$$

where $s = \pm 1$. These yield four distinct families of constraint solutions. This simplified form of the matching constraints was pointed out in an article by Hanquin and Demaret.[15]

The "bit of calculation" mentioned above is actually quite an ordeal. There is a more elegant way to find the restrictions on the wave amplitudes. In a recent paper,[11] it was pointed out that the constraint equations lead to an expression involving only the Wronskian of the Legendre polynomials. The use of Abel's identity then yields the constraints given in 91 and 92.

Note that the matching constraints do not permit solutions in which all of the coefficients C_n are zero. Because these are the coefficients of terms that diverge logarithmically near the initial singularity, it is guaranteed that these terms will always dominate the asymptotic behavior of these solutions.

The imposition of the matching constraints ensures the geometry is regular along the null hypersurfaces, but unless further constraints are imposed, we will have conical singularities along the symmetry axes.[14] To avoid a conical singularity at $\theta = 0$, the metric function a must obey the equation

$$\left.\frac{\partial a}{\partial \theta}\right|_{\theta=0} = 0. \tag{93}$$

The ratio of the coefficients of $d\theta^2$ and $\theta^2 d\delta^2$ in the form of the three-metric[5] asymptotic to $\theta = 0$ must equal its flat space value of unity. This is a constraint on the metric functions a and ψ that can be written as

$$(a + W)|_{\theta=0} = \frac{1}{2} \ln \left(\frac{1}{2} \sin t \right) \tag{94}$$

where ψ has been written in terms of W via equation 83. Similarly, at $\theta = \pi$, the metric function a must obey the equation

$$\left.\frac{\partial a}{\partial \theta}\right|_{\theta=\pi} = 0, \tag{95}$$

and consideration of the ratio of the coefficients of $d\theta^2$ and $\theta^2 d\sigma^2$ in the form of the three-metric asymptotic to $\theta = \pi$ yields a second nonconicality constraint:

$$(a - W)|_{\theta=\pi} = \frac{1}{2} \ln \left(\frac{1}{2} \sin t \right). \tag{96}$$

The constraints 94 and 96 ensure regularity of the geometry of the constant t hypersurfaces. These are called the primary nonconicality constraints. To ensure the geometry of the full spacetime is regular, we must impose a set of secondary

[5] This is the metric of the three-dimensional hypersurfaces. While the coordinates of a hypersurface could, in principle, contain the time coordinate, the term is usually reserved for hypersurfaces whose components are spatial. In this thesis, a three-metric will always refer to spacelike hypersurfaces.

nonconicality constraints:[6]

$$\left.\left(\frac{\partial a}{\partial t} + \frac{\partial W}{\partial t}\right)\right|_{\theta=0} = \frac{1}{2}\cot t, \tag{97}$$

and

$$\left.\left(\frac{\partial a}{\partial t} - \frac{\partial W}{\partial t}\right)\right|_{\theta=\pi} = \frac{1}{2}\cot t. \tag{98}$$

[6] From the unpublished notes of Robert H. Gowdy.

Chapter 5
Cosmic Censorship and Asymptotic Behavior in Gowdy Spacetimes

Gowdy's solutions to Einstein's field equations have a spacelike singularity at $t = 0$, and those with the topologies of the wormhole and the three-sphere contain an additional spacelike singularity at $t = \pi$. That cosmologies generally have singularities is a consequence of the Penrose-Hawking singularity theorems.[16] However, these theorems say nothing about the nature of the singularities. To understand causal structure, it is necessary to know something about the singularities in the spacetime.

5.1 Cosmic Censorship in Gowdy Spacetimes

The cosmic censorship conjecture, "nature abhors a naked singularity," was discussed in the introduction. While spacetimes with naked singularities have been constructed, they are generally dismissed as being physically unrealistic.[1] It is important to search for causality violation in more realistic spacetimes. As mentioned earlier, Gowdy universes have boundary conditions that are considered cosmologically reasonable. The wormhole universes describe the interior of black holes, and since the strong cosmic censorship conjecture rules out the possibility of naked singularities even in black holes, these have physical significance in the search for causality violation.

In the case of polarized Gowdy spacetimes, strong cosmic censorship appears to hold.[6] However, there is reason to believe a naked singularity may appear in unpolarized spacetimes with the wormhole or three-sphere topologies. As mentioned earlier, the gravitational waves in these spacetimes are locally Einstein-Rosen cylindrical waves. If these waves were to focus down onto the symmetry axis, infinite curvature may occur.

5.2 Asymptotic behavior of the metric

It has been conjectured that, in most cosmological solutions, the evolution equations of the spacetime are asymptotically velocity term dominated. This conjecture, known as the BKL conjecture after its authors Belinski, Khalatnikov, and Lifschitz, is, in essence, the statement that, near the singularity, terms in the evolution equations involving time derivatives will dominate terms involving only spatial derivatives. In other words, spatial derivatives of the metric functions can be dropped from the evolution equations. A version of this property given by Isenberg and Moncrief,[6] known as the asymptotically-velocity-term-dominated-near-the-singularity (AVTDS) property, was proven to hold in polarized Gowdy spacetimes except, perhaps, near the symmetry axes. To show that the AVTDS property holds, it is sufficient to show the metric is locally that of the Kasner universe as the singularity is approached.

Numerical studies corroborate the work of Isenberg and Moncrief.[17] However, in generic Gowdy spacetimes, there are two polarizations to the gravitational waves. If both polarizations are left in, the problem is far more difficult. Numerical evidence has not concluded the AVTDS property holds in generic G_2 spacetimes. In the numerical analysis, spikey features appear, and these are not well understood.[18] It is hoped that a better understanding of these features and the reasons they appear may lead to more conclusive results. However, at least in the case of the wormhole topology, it appears the singularity is AVTDS in regions where the spikey features do not occur.[19] It is not expected that AVTDS will hold in the generic case. The behavior near the singularity may be that of the homogeneous Mixmaster type rather than Kasner-like.[20]

Chapter 6
Geodesic Behavior in Gowdy S^3 Spacetimes

In this chapter, we consider null (or lightlike) geodesics on the symmetry axes $\theta = 0$ and $\theta = \pi$. We want to find expressions for $\frac{d\sigma}{dt}$ and $\frac{d\delta}{dt}$ near the initial and final singularities of the spacetime for each of the four families of solutions.

The null geodesic constraint equation yields

$$\left(\frac{dt}{d\lambda}\right)^2 = \left(\frac{d\theta}{d\lambda}\right)^2 + \left(\frac{d\sigma}{d\lambda}\right)^2 Re^{2(\psi-a)} + \left(\frac{d\delta}{d\lambda}\right)^2 Re^{-2(\psi+a)} \tag{99}$$

The geodesics are restricted to the symmetry axes, so $\frac{d\theta}{d\lambda}$ vanishes for both axes. On the $\theta = 0$ axis, $\frac{d\delta}{d\lambda}$ vanishes, and equation 99 reduces to

$$\left.\left(\frac{dt}{d\lambda}\right)^2\right|_{\theta=0} = \left.\left(\frac{d\sigma}{d\lambda}\right)^2 Re^{2(\psi-a)}\right|_{\theta=0} \tag{100}$$

Multiplying both sides by $\left(\frac{d\lambda}{dt}\right)^2$ yields

$$1 = \left(\frac{d\lambda}{dt}\frac{d\sigma}{d\lambda}\right)^2 Re^{2(\psi-a)} \tag{101}$$

By the chain rule, $\frac{d\sigma}{dt} = \frac{d\sigma}{d\lambda}\frac{d\lambda}{dt}$, so

$$1 = \left(\frac{d\sigma}{dt}\right)^2 Re^{2(\psi-a)} \tag{102}$$

Solving for $\frac{d\sigma}{dt}$ yields

$$\left.\frac{d\sigma}{dt}\right|_{\theta=0} = \left.\frac{1}{\sqrt{R}}e^{(a-\psi)}\right|_{\theta=0} \tag{103}$$

From the nonconicality constraints we find

$$a(0,t) = \frac{1}{2}\ln\left(\frac{1}{2}\sin t\right) - W(1,\cos t) \tag{104}$$

where

$$W\left(1,\cos t\right)=\sum_{\ell}\left[A_{\ell}P_{\ell}\left(\cos t\right)+C_{\ell}Q_{\ell}\left(\cos t\right)\right] \tag{105}$$

For the $\mathbb{R}\times S^3$ spacetimes,

$$R=\sin\theta\sin t \tag{106}$$

and

$$\psi=W-\frac{1}{2}\ln\tan\left(\frac{\theta}{2}\right) \tag{107}$$

Equation 103 becomes

$$\begin{aligned}\frac{d\sigma}{dt} &= \frac{1}{\sqrt{\sin\theta\sin t}}e^{(1/2)\ln((1/2)\sin t)-2W(1,\cos t)+(1/2)\ln\tan(\theta/2)} && (108)\\ &= \frac{1}{\sqrt{\sin\theta\sin t}}\sqrt{(1/2)\sin t\,\tan\left(\theta/2\right)}e^{-2W(1,\cos t)} && (109)\\ &= \frac{1}{\sqrt{2\sin\left(\theta/2\right)\cos\left(\theta/2\right)\sin t}}\sqrt{(1/2)\sin t\,\tan\left(\theta/2\right)}e^{-2W(1,\cos t)} && (110)\\ &= \frac{1}{\sqrt{4\cos^2\left(\theta/2\right)}}e^{-2W(1,\cos t)} && (111)\end{aligned}$$

Evaluating at $\theta=0$ yields

$$\left.\frac{d\sigma}{dt}\right|_{\theta=0}=\frac{1}{2}e^{-2W(1,\cos t)} \tag{112}$$

We now turn to the function $W\left(1,\cos t\right)$. Recall that the Legendre functions of the second kind are given by the series solution[8]

$$Q_{\ell}=\frac{1}{2}\ln\left(\frac{1+\cos t}{1-\cos t}\right)P_{\ell}\left(\cos t\right)-B_{\ell}\left(\cos t\right) \tag{113}$$

where

$$B_{\ell}\left(x\right)=\frac{2n-1}{n}P_{n-1}\left(x\right)+\frac{2n-5}{3\left(n-1\right)}P_{n-3}\left(x\right)+\cdots. \tag{114}$$

Splitting the Q_ℓ into even and odd functions yields

$$W(1,\tau) = \sum_\ell \left[A_\ell P_\ell(\tau) - C_\ell B_\ell(\tau) + \frac{1}{2}\ln\left(\frac{1+\tau}{1-\tau}\right)\left(C_{2\ell}P_{2\ell}(\tau) + C_{2\ell+1}P_{2\ell+1}(\tau)\right)\right] \tag{115}$$

where $\tau = \cos t$. Substituting this equation into equation 112 yields

$$\left.\frac{d\sigma}{dt}\right|_{\theta=0} = \frac{1}{2}\left(\frac{1+\tau}{1-\tau}\right)^{-\sum_\ell [C_{2\ell}P_{2\ell}(\tau)+C_{2\ell+1}P_{2\ell+1}(\tau)]} e^{-2\sum_\ell [A_\ell P_\ell(\tau) - C_\ell B_\ell(\tau)]} \tag{116}$$

On the $\theta = \pi$ axis, $\frac{d\sigma}{d\lambda}$ vanishes, and equation 99 reduces to

$$\left.\left(\frac{dt}{d\lambda}\right)^2\right|_{\theta=\pi} = \left.\left(\frac{d\delta}{d\lambda}\right)^2 Re^{-2(\psi+a)}\right|_{\theta=\pi} \tag{117}$$

Solving for $\frac{d\delta}{dt}$ yields

$$\left.\frac{d\delta}{dt}\right|_{\theta=\pi} = \left.\frac{1}{\sqrt{R}}e^{(a+\psi)}\right|_{\theta=\pi} \tag{118}$$

The nonconicality constraint for the $\theta = \pi$ axis gives

$$a(\pi,t) = \frac{1}{2}\ln\left(\frac{1}{2}\sin t\right) + W(-1,\tau) \tag{119}$$

where

$$W(-1,\tau) = \sum_\ell \left[(-1)^\ell A_\ell P_\ell(\tau) + (-1)^\ell C_\ell Q_\ell(\tau)\right] \tag{120}$$

A procedure analogous to that used to find equation 116 yields

$$\left.\frac{d\delta}{dt}\right|_{\theta=\pi} = \frac{1}{2}\left(\frac{1+\tau}{1-\tau}\right)^{\sum_\ell [C_{2\ell}P_{2\ell}(\tau)-C_{2\ell+1}P_{2\ell+1}(\tau)]} e^{2\sum_\ell \left[(-1)^\ell A_\ell P_\ell(\tau) - C_{2\ell}B_{2\ell}(\tau) + C_{2\ell+1}B_{2\ell+1}(\tau)\right]} \tag{121}$$

A photon approaching the initial singularity will follow a path described by the limits of equations 116 and 121 as τ approaches 1 (*i.e.* as t approaches zero). Taking the limits yields

$$\lim_{\tau\to 1}\left.\frac{d\sigma}{dt}\right|_{\theta=0} = \frac{1}{2}\lim_{\tau\to 1}\left(\frac{1+\tau}{1-\tau}\right)^{-\sum_\ell [C_{2\ell}+C_{2\ell+1}]} e^{-2\sum_\ell [A_\ell - C_\ell B_\ell(1)]} \tag{122}$$

and

$$\lim_{\tau\to 1}\left.\frac{d\delta}{dt}\right|_{\theta=\pi} = \frac{1}{2}\lim_{\tau\to 1}\left(\frac{1+\tau}{1-\tau}\right)^{\sum_\ell [C_{2\ell}-C_{2\ell+1}]} e^{2\sum_\ell \left[(-1)^\ell A_\ell - C_{2\ell}B_{2\ell}(1)+C_{2\ell+1}B_{2\ell+1}(1)\right]} \tag{123}$$

The matching constraints imposed restrictions on the wave amplitudes C_ℓ yielding four distinct families of solutions. The restrictions are recalled here for convenience.

$$\begin{array}{llll} \text{Type I:} & \sum_{\ell=0}^{\infty} C_{2\ell} = 0 & \text{and} & \sum_{\ell=0}^{\infty} C_{2\ell+1} = s + \frac{1}{2} \\ \text{Type II:} & \sum_{\ell=0}^{\infty} C_{2\ell} = s & \text{and} & \sum_{\ell=0}^{\infty} C_{2\ell+1} = \frac{1}{2} \end{array} \tag{124}$$

where $s = \pm 1$. We use these constraints to get the following solutions near the initial singularity.

$$\begin{array}{ll} \text{Type I}_+ : & \left.\frac{d\sigma}{dt}\right|_{\theta=0} = \frac{1}{2}\lim_{\tau\to 1}\left(\frac{1+\tau}{1-\tau}\right)^{-3/2} e^{-2\sum_\ell [A_\ell - C_\ell B_\ell(1)]} \\ & \left.\frac{d\delta}{dt}\right|_{\theta=\pi} = \frac{1}{2}\lim_{\tau\to 1}\left(\frac{1+\tau}{1-\tau}\right)^{-3/2} e^{2\sum_\ell \left[(-1)^\ell A_\ell - C_{2\ell}B_{2\ell}(1)+C_{2\ell+1}B_{2\ell+1}(1)\right]} \\ \text{Type I}_- : & \left.\frac{d\sigma}{dt}\right|_{\theta=0} = \frac{1}{2}\lim_{\tau\to 1}\left(\frac{1+\tau}{1-\tau}\right)^{1/2} e^{-2\sum_\ell [A_\ell - C_\ell B_\ell(1)]} \\ & \left.\frac{d\delta}{dt}\right|_{\theta=\pi} = \frac{1}{2}\lim_{\tau\to 1}\left(\frac{1+\tau}{1-\tau}\right)^{1/2} e^{2\sum_\ell \left[(-1)^\ell A_\ell - C_{2\ell}B_{2\ell}(1)+C_{2\ell+1}B_{2\ell+1}(1)\right]} \\ \text{Type II}_+ : & \left.\frac{d\sigma}{dt}\right|_{\theta=0} = \frac{1}{2}\lim_{\tau\to 1}\left(\frac{1+\tau}{1-\tau}\right)^{-3/2} e^{-2\sum_\ell [A_\ell - C_\ell B_\ell(1)]} \\ & \left.\frac{d\delta}{dt}\right|_{\theta=\pi} = \frac{1}{2}\lim_{\tau\to 1}\left(\frac{1+\tau}{1-\tau}\right)^{1/2} e^{2\sum_\ell \left[(-1)^\ell A_\ell - C_{2\ell}B_{2\ell}(1)+C_{2\ell+1}B_{2\ell+1}(1)\right]} \\ \text{Type II}_- : & \left.\frac{d\sigma}{dt}\right|_{\theta=0} = \frac{1}{2}\lim_{\tau\to 1}\left(\frac{1+\tau}{1-\tau}\right)^{1/2} e^{-2\sum_\ell [A_\ell - C_\ell B_\ell(1)]} \\ & \left.\frac{d\delta}{dt}\right|_{\theta=\pi} = \frac{1}{2}\lim_{\tau\to 1}\left(\frac{1+\tau}{1-\tau}\right)^{-3/2} e^{2\sum_\ell \left[(-1)^\ell A_\ell - C_{2\ell}B_{2\ell}(1)+C_{2\ell+1}B_{2\ell+1}(1)\right]} \end{array} \tag{125}$$

The subscript + or $-$ corresponds to the sign choice for s.

To find the solutions corresponding to a photon approaching the final singularity, we consider the limits of equations 116 and 121 as τ approaches -1 (*i.e.* as t

approaches π). The procedure is analogous to that above. We find that

$$\lim_{\tau\to-1} \left.\frac{d\sigma}{dt}\right|_{\theta=0} = \frac{1}{2}\lim_{\tau\to-1}\left(\frac{1+\tau}{1-\tau}\right)^{\sum_\ell [C_{2\ell+1}-C_{2\ell}]} e^{-2\sum_\ell\left[(-1)^\ell A_\ell - C_\ell B_\ell(-1)\right]} \tag{126}$$

and

$$\lim_{\tau\to-1} \left.\frac{d\delta}{dt}\right|_{\theta=\pi} = \frac{1}{2}\lim_{\tau\to-1}\left(\frac{1+\tau}{1-\tau}\right)^{\sum_\ell [C_{2\ell}+C_{2\ell+1}]} e^{2\sum_\ell [A_\ell - C_{2\ell}B_{2\ell}(-1)+C_{2\ell+1}B_{2\ell+1}(-1)]} \tag{127}$$

Using the matching constraints, we get the following solutions near the final singularity.

$$\begin{aligned}
\text{Type I}_+ :\quad & \left.\tfrac{d\sigma}{dt}\right|_{\theta=0} = \tfrac{1}{2}\lim_{\tau\to-1}\left(\tfrac{1+\tau}{1-\tau}\right)^{3/2} e^{-2\sum_\ell [A_\ell - C_\ell B_\ell(-1)]} \\
& \left.\tfrac{d\delta}{dt}\right|_{\theta=\pi} = \tfrac{1}{2}\lim_{\tau\to-1}\left(\tfrac{1+\tau}{1-\tau}\right)^{3/2} e^{2\sum_\ell [A_\ell - C_{2\ell}B_{2\ell}(-1)+C_{2\ell+1}B_{2\ell+1}(-1)]} \\
\text{Type I}_- :\quad & \left.\tfrac{d\sigma}{dt}\right|_{\theta=0} = \tfrac{1}{2}\lim_{\tau\to-1}\left(\tfrac{1+\tau}{1-\tau}\right)^{-1/2} e^{-2\sum_\ell [A_\ell - C_\ell B_\ell(-1)]} \\
& \left.\tfrac{d\delta}{dt}\right|_{\theta=\pi} = \tfrac{1}{2}\lim_{\tau\to-1}\left(\tfrac{1+\tau}{1-\tau}\right)^{-1/2} e^{2\sum_\ell [A_\ell - C_{2\ell}B_{2\ell}(-1)+C_{2\ell+1}B_{2\ell+1}(-1)]} \\
\text{Type II}_+ :\quad & \left.\tfrac{d\sigma}{dt}\right|_{\theta=0} = \tfrac{1}{2}\lim_{\tau\to-1}\left(\tfrac{1+\tau}{1-\tau}\right)^{-1/2} e^{-2\sum_\ell [A_\ell - C_\ell B_\ell(-1)]} \\
& \left.\tfrac{d\delta}{dt}\right|_{\theta=\pi} = \tfrac{1}{2}\lim_{\tau\to-1}\left(\tfrac{1+\tau}{1-\tau}\right)^{3/2} e^{2\sum_\ell [A_\ell - C_{2\ell}B_{2\ell}(-1)+C_{2\ell+1}B_{2\ell+1}(-1)]} \\
\text{Type II}_- :\quad & \left.\tfrac{d\sigma}{dt}\right|_{\theta=0} = \tfrac{1}{2}\lim_{\tau\to-1}\left(\tfrac{1+\tau}{1-\tau}\right)^{3/2} e^{-2\sum_\ell [A_\ell - C_\ell B_\ell(-1)]} \\
& \left.\tfrac{d\delta}{dt}\right|_{\theta=\pi} = \tfrac{1}{2}\lim_{\tau\to-1}\left(\tfrac{1+\tau}{1-\tau}\right)^{-1/2} e^{2\sum_\ell [A_\ell - C_{2\ell}B_{2\ell}(-1)+C_{2\ell+1}B_{2\ell+1}(-1)]}
\end{aligned} \tag{128}$$

From the solutions given above for the time rate of change of the coordinates σ and δ as we head backward toward the initial singularity (or forward toward the final singularity), along a symmetry axis, we see that, in some cases, a photon will wind around the entire universe infinitely many times. In other words, near the initial singularity, the past lightcone spreads out and every part of the early universe is causally connected to every other part. Similarly, near the final singularity, the future lightcone spreads out and information from a single event can reach every part of the universe. The results for the initial singularity are more interesting, because, in the case of the final singularity, very shortly after the future lightcones spread out so that they all intersect, the universe will end in a big crunch.

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